

MULTILINEAR FRACTIONAL INTEGRAL OPERATORS ON NON-HOMOGENEOUS METRIC MEASURE SPACES

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ABSTRACT. Let (X, d, μ) be a non-homogeneous metric measure space satisfying both the geometrically doubling and the upper doubling measure conditions. In this paper, the boundedness of multilinear fractional integral operator in this setting is proved. Via a sharp maximal operator, the boundedness of commutators generated by multilinear fractional integral operator with $RBMO(\mu)$ function on non-homogeneous metric measure spaces in Lebesgue spaces is obtained.

1. INTRODUCTION AND PRELIMINARIES

It is well known that a space of homogeneous type is the space, which satisfies the assumption of the doubling measure condition, i.e. there exists a constant $C > 0$ such that $\mu(B(x, 2r)) \leq C\mu(B(x, r))$ for all $x \in \text{supp}\mu$ and $r > 0$. However non-doubling measure is a nonnegative measure μ only satisfies the polynomial growth condition, i.e., for all $x \in X$ and $r > 0$, there exists a constant $C > 0$ and $k \in (0, n]$ such that,

$$\mu(B(x, r)) \leq C_0 r^k, \quad (1.1)$$

where $B(x, r) = \{y \in X : |y - x| < r\}$. This brings rapid development in harmonic analysis (see [2, 7, 10, 11, 22, 24, 25, 27]). As an important application, it is to solve the long-standing open Painlevé's problem (see [24]).

In [13], Hytönen pointed out that the doubling measure is not the special case of the non-doubling measures. To overcome this difficulty, a kind of metric measure space (X, d, μ) , that satisfies the geometrically doubling and the upper doubling measure conditions (see Definition 1.1 and 1.2) is introduced by Hytönen in [13], which is called the non-homogeneous metric measure space. The highlight of this kind of space is that it includes both space of homogeneous type and metric spaces with polynomial growth measures as special cases. From then on, a lot of results paralleled to homogeneous spaces and non-doubling measure spaces are obtained (see [1, 4, 5, 6, 13, 14, 15, 16, 18, 19, 20, 21] and the references therein). For example, Hytönen et al. [16] and Bui and Duong[1] independently introduced the atomic Hardy space $\mathcal{H}^1(\mu)$ and obtained that the dual space of $\mathcal{H}^1(\mu)$ is $RBMO(\mu)$. Bui and Duong[1] also proved that Calderón-Zygmund operator and commutators of Calderón-Zygmund operator with RBMO function are bounded

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in $L^p(\mu)$ for $1 < p < \infty$. Later, Lin and Yang [18] introduced the space $RBLO(\mu)$ and proved the maximal Calderón-Zygmund operator is bounded from $L^\infty(\mu)$ into $RBLO(\mu)$. Recently, some equivalent characterizations was established by Liu et al. [21] for the boundedness of Carderon-Zygmund operators on $L^p(\mu)$ for $1 < p < \infty$. Fu et al. [5, 6] established the boundedness of multilinear commutators of Calderon-Zygmund operators and commutators of generalized fractional integrals with $RBMO(\mu)$. Fu et al. [4] partially established the theory of the Hardy space \mathcal{H}^p with $p \in (0, 1]$ on (X, d, μ) . The readers can refer to the survey [30] and the monograph [31] for more developments on harmonic analysis in non-homogeneous metric measure spaces.

At the other hand, the theory on multilinear integral operators has been studied by some researchers. Coifman and Meyers[3] firstly established the theory of bilinear Calderón-Zygmund operators. Later, Gorafakos and Torres [8, 9] established the boundedness of multilinear singular integral on the product Lebesgue spaces and Hardy spaces. Xu[28, 29] established the properties of multilinear singular integrals and commutators on non-doubling measure spaces (R^n, μ) . The bounedness of multilinear fractional integral and commutators on non-doubling measure spaces (R^n, μ) was proved by Lian and Wu in [17]. In non-homogeneous metric measure spaces, Hu et al. [12] established the weighted norm inequalities for multilinear Calderón-Zygmund operators. The boundedness of commutators of multilinear singular integrals on Lebesgue spaces was obtained by Xie et al. in [26].

In this paper, multilinear fractional integral operator and commutators generated by multilinear fractional integral with $RBMO(\mu)$ function on non-homogeneous metric spaces are introduced. And it is proved that multilinear farctional integral operators and commutators are bounded in Lebesgue spaces on non-homogeneous metric spaces, provided that factional integral is bounded from L^r into L^s for all $r \in (1, 1/\beta)$ and $1/s = 1/r - \beta$ with $0 < \beta < 1$. The results in this paper include the corresponding results on both the homogeneous spaces and (R^n, μ) with non-doubling measure spaces.

We first recall some notations and definitions.

Definition 1.1. [13] A metric spaces (X, d) is called geometrically doubling if there exists some $N_0 \in \mathbf{N}$ such that, for any ball $B(x, r) \subset X$, there exists a finite ball covering $\{B(x_i, r/2)\}_i$ of $B(x, r)$ such that the cardinality of this covering is at most N_0 .

Definition 1.2. [13] A metric measure space (X, d, μ) is said to be upper doubling if μ is a Borel measure on X and there exists a function $\lambda : X \times (0, +\infty) \rightarrow (0, +\infty)$ and a constant $C_\lambda > 0$ such that for each $x \in X, r \mapsto (x, r)$ is non-decreasing, and for all $x \in X, r > 0$,

$$\mu(B(x, r)) \leq \lambda(x, r) \leq C_\lambda \lambda(x, r/2). \quad (1.2)$$

Remark 1.3. (i) A space of homogeneous type is a special case of upper doubling spaces, where one can take $\lambda(x, r) \equiv \mu(B(x, r))$. On the other hand, a metric space (X, d, μ) satisfying the polynomial growth condition (1.1)(in particular,

$(X, d, \mu) \equiv (R^n, |\cdot|, \mu)$ with μ satisfying (1.1) for some $k \in (0, n]$) is also an upper doubling measure space if we take $\lambda(x, r) \equiv Cr^k$.

(ii) Let (X, d, μ) be an upper doubling space and λ be a function on $X \times (0, +\infty)$ as in Definition 1.2. In [13], it was showed that there exists another function $\tilde{\lambda}$ such that for all $x, y \in X$ with $d(x, y) \leq r$,

$$\tilde{\lambda}(x, r) \leq \tilde{C}\tilde{\lambda}(y, r). \quad (1.3)$$

Thus, in this paper, we always suppose that λ satisfies (1.3) and $\lambda(x, ar) \geq a^m \lambda(x, r)$ for all $x \in X$ and $a, r > 0$.

(iii) As shown in [23], the upper doubling condition is equivalent to the weak growth condition: there exists a function $\lambda : X \times (0, \infty) \rightarrow (0, \infty)$, with $r \rightarrow \lambda(x, r)$ non-decreasing, a positive constant C_λ depending on λ and ϵ such that

(a) for all $r \in (0, \infty)$, $t \in [0, r]$, $x, y \in X$ and $d(x, y) \in [0, r]$,

$$|\lambda(y, r+t) - \lambda(x, r)| \leq C_\lambda \left[\frac{d(x, y) + t}{r} \right]^\epsilon \lambda(x, r);$$

(b) for all $x \in X$ and $r \in (0, \infty)$,

$$\mu(B(x, r)) \leq \lambda(x, r).$$

Definition 1.4. Let $\alpha, \beta \in (1, +\infty)$ and a ball $B \subset X$ is called (α, β) -doubling if $\mu(\alpha B) \leq \beta \mu(B)$.

As in Lemma 2.3 of [1], there exist plenty of doubling balls with small radii and with large radii. Throughout this paper, unless α and β are specified otherwise, by an (α, β) doubling ball we mean a $(6, \beta_0)$ -doubling with a fixed number $\beta_0 > \max\{C_\lambda^{3 \log_2 6}, 6^n\}$, where $n = \log_2 N_0$ be viewed as a geometric dimension of the spaces.

Definition 1.5. [6] Let $0 \leq \gamma < 1$. For any two balls $B \subset Q$, set $N_{B,Q}$ be the smallest integer satisfying $6^{N_{B,Q}} r_B \geq r_Q$, then one defines

$$K_{B,Q}^{(\gamma)} = 1 + \sum_{k=1}^{N_{B,Q}} \left[\frac{\mu(6^k B)}{\lambda(x_B, 6^k r_B)} \right]^{(1-\gamma)}. \quad (1.4)$$

For $\gamma = 0$, we simply write $K_{B,Q}^{(0)} = K_{B,Q}$.

The multilinear fractional integral on nonhomogeneous metric measure spaces is defined as follows.

Definition 1.6. Let $\alpha \in (0, m)$. A kernel

$$K(\cdot, \dots, \cdot) \in L_{loc}^1((X)^{m+1} \setminus \{(x, y_1, \dots, y_j, \dots, y_m) : x = y_j, 1 \leq j \leq m\})$$

is called an m -linear fractional integral kernel if it satisfies:

(i)

$$|K(x, y_1, \dots, y_j, \dots, y_m)| \leq \frac{C}{\left[\sum_{j=1}^m \lambda(x, d(x, y_j)) \right]^{m-\alpha}} \quad (1.5)$$

for all $(x, y_1, \dots, y_j, \dots, y_m) \in (X)^{m+1}$ with $x \neq y_j$ for some j .

(ii) There exists $0 < \delta \leq 1$ such that

$$\begin{aligned} & |K(x, y_1, \dots, y_j, \dots, y_m) - K(x', y_1, \dots, y_j, \dots, y_m)| \\ & \leq \frac{Cd(x, x')^\delta}{\left[\sum_{j=1}^m d(x, y_j) \right]^\delta \left[\sum_{j=1}^m \lambda(x, d(x, y_j)) \right]^{m-\alpha}}, \end{aligned} \quad (1.6)$$

proved that $Cd(x, x') \leq \max_{1 \leq j \leq m} d(x, y_j)$ and for each j ,

$$\begin{aligned} & |K(x, y_1, \dots, y_j, \dots, y_m) - K(x, y_1, \dots, y'_j, \dots, y_m)| \\ & \leq \frac{Cd(y_j, y'_j)^\delta}{\left[\sum_{j=1}^m d(x, y_j) \right]^\delta \left[\sum_{j=1}^m \lambda(x, d(x, y_j)) \right]^{m-\alpha}}, \end{aligned} \quad (1.7)$$

proved that $Cd(y_j, y'_j) \leq \max_{1 \leq j \leq m} d(x, y_j)$.

A multilinear operators $I_{\alpha, m}$ is called the multilinear fractional integral operator with the above kernel K satisfying (1.5), (1.6) and (1.7) if, for f_1, \dots, f_m are L^∞ functions with compact support and $x \notin \bigcap_{j=1}^m \text{supp} f_j$,

$$\begin{aligned} & I_{\alpha, m}(f_1, \dots, f_m)(x) \\ & = \int_{X^m} K(x, y_1, \dots, y_m) f_1(y_1) \cdots f_m(y_m) d\mu(y_1) \cdots d\mu(y_m). \end{aligned} \quad (1.8)$$

For $m = 1$, we simply write $I_{\alpha, 1}$ by I_α , which is the generalized fractional integral operator introduced by [6].

Remark 1.7. Because $\max_{1 \leq j \leq m} d(x, y_j) \leq \sum_{j=1}^m d(x, y_j) \leq m \max_{1 \leq j \leq m} d(x, y_j)$, (ii) in

Definition 1.5 is equivalent to (ii') in the following statement.

(ii') There exists $0 < \delta \leq 1$ such that

$$\begin{aligned} & |K(x, y_1, \dots, y_j, \dots, y_m) - K(x', y_1, \dots, y_j, \dots, y_m)| \\ & \leq \frac{Cd(x, x')^\delta}{\left[\max_{1 \leq j \leq m} d(x, y_j) \right]^\delta \left[\sum_{j=1}^m \lambda(x, d(x, y_j)) \right]^{m-\alpha}}, \end{aligned}$$

proved that $Cd(x, x') \leq \max_{1 \leq j \leq m} d(x, y_j)$ and for each j ,

$$\begin{aligned} & |K(x, y_1, \dots, y_j, \dots, y_m) - K(x, y_1, \dots, y'_j, \dots, y_m)| \\ & \leq \frac{Cd(y_j, y'_j)^\delta}{\left[\max_{1 \leq j \leq m} d(x, y_j) \right]^\delta \left[\sum_{j=1}^m \lambda(x, d(x, y_j)) \right]^{m-\alpha}}, \end{aligned}$$

proved that $Cd(y_j, y'_j) \leq \max_{1 \leq j \leq m} d(x, y_j)$.

Definition 1.8. [1] Let $\rho > 1$ be some constant. A function $b \in L^1_{loc}(\mu)$ is said to belong to $RBMO(\mu)$ if there exists a constant $C > 0$ such that for any ball B

$$\frac{1}{\mu(\rho B)} \int_B |b(x) - m_{\tilde{B}} b| d\mu(x) \leq C, \quad (1.9)$$

and for any two doubling ball $B \subset Q$,

$$|m_B(b) - m_Q(b)| \leq CK_{B,Q}. \quad (1.10)$$

\tilde{B} is the smallest (α, β) -doubling ball of the form $6^k B$ with $k \in \mathbf{N} \cup \{0\}$, and $m_{\tilde{B}}(b)$ is the mean value of b on \tilde{B} , namely,

$$m_{\tilde{B}}(b) = \frac{1}{\mu(\tilde{B})} \int_{\tilde{B}} b(x) d\mu(x).$$

The minimal constant C appearing in (1.9) and (1.10) is defined to be the $RBMO(\mu)$ norm of f and denoted by $\|b\|_*$.

For $1 \leq i \leq m$, denote by C_i^m the family of all finite subsets $\sigma = \{\sigma(1), \sigma(2), \dots, \sigma(i)\}$ of $\{1, 2, \dots, m\}$ with i different elements. For any $\sigma \in C_i^m$, the complementary sequences σ' is given by $\sigma' = \{1, 2, \dots, m\} \setminus \sigma$. Moreover, for $b_i \in RBMO(\mu)$, $i = 1, \dots, m$, let $\vec{b} = (b_1, b_2, \dots, b_m)$ be a finite family of locally integrable functions. For all $1 \leq i \leq m$ and $\sigma = \{\sigma(1), \dots, \sigma(i)\} \in C_i^m$, we set $\vec{b}_\sigma = (b_{\sigma(1)}, \dots, b_{\sigma(i)})$ and the product $b_\sigma(x) = b_{\sigma(1)}(x) \cdots b_{\sigma(i)}(x)$. Also, we denote $\vec{f} = (f_1, \dots, f_m)$, $\vec{f}_\sigma = (f_{\sigma(1)}, \dots, f_{\sigma(i)})$ and $\vec{b}_{\sigma'} \vec{f}_{\sigma'} = (b_{\sigma'(i+1)} f_{\sigma'(i+1)}, \dots, b_{\sigma'(m)} f_{\sigma'(m)})$.

Definition 1.9. A kind of commutators generated by multilinear fractional integral operators $I_{\alpha,m}$ with $b_i \in RBMO(\mu)$, $i = 1, \dots, m$ is defined as follows:

$$[\vec{b}, I_{\alpha,m}](\vec{f})(x) = \sum_{i=0}^m \sum_{\sigma \in C_i^m} (-1)^{m-i} b_\sigma(x) I_{\alpha,m}(\vec{f}_\sigma, \vec{b}_{\sigma'} \vec{f}_{\sigma'})(x).$$

In particular, when $m = 2$, it is easy to see that

$$\begin{aligned} [b_1, b_2, I_{\alpha,2}](f_1, f_2)(x) &= b_1(x) b_2(x) I_{\alpha,2}(f_1, f_2)(x) - b_1(x) I_{\alpha,2}(f_1, b_2 f_2)(x) \\ &\quad - b_2(x) I_{\alpha,2}(b_1 f_1, f_2)(x) + I_{\alpha,2}(b_1 f_1, b_2 f_2)(x). \end{aligned} \quad (1.11)$$

$[b_1, I_{\alpha,2}]$ and $[b_2, I_{\alpha,2}]$ are defined as follows respectively.

$$\begin{aligned} [b_1, I_{\alpha,2}](f_1, f_2)(x) &= b_1(x) I_{\alpha,2}(f_1, f_2)(x) - I_{\alpha,2}(b_1 f_1, f_2)(x), \\ [b_2, I_{\alpha,2}](f_1, f_2)(x) &= b_2(x) I_{\alpha,2}(f_1, f_2)(x) - I_{\alpha,2}(f_1, b_2 f_2)(x). \end{aligned}$$

Without loss of generality, we only consider the case of $m = 2$. Now let us state the main results.

Theorem 1.10. Let $0 < \alpha < 2$, $1 < p_1, p_2 < +\infty$, $f_1 \in L^{p_1}(\mu)$ and $f_2 \in L^{p_2}(\mu)$. If I_β is bounded from L^r into L^s for all $r \in (1, 1/\beta)$ and $1/s = 1/r - \beta$ with $0 < \beta < 1$, then there exists a constant $C > 0$ such that

$$\|I_{\alpha,2}(f_1, f_2)\|_{L^q(\mu)} \leq C \|f_1\|_{L^{p_1}(\mu)} \|f_2\|_{L^{p_2}(\mu)},$$

where $\frac{1}{q} = \frac{1}{p_1} + \frac{1}{p_2} - \alpha$.

Theorem 1.11. *Set μ is a Radon measure with $\|\mu\| = \infty$, $0 < \alpha < 2$, $1 < p_1$, $p_2 < +\infty$, $f_1 \in L^{p_1}(\mu)$, $f_2 \in L^{p_2}(\mu)$, $b_1, b_2 \in RBMO(\mu)$ and if I_β is bounded from L^r into L^s for any $r \in (1, 1/\beta)$, $1/s = 1/r - \beta$ with $0 < \beta < 1$, then there exists a constant $C > 0$ such that*

$$\|[b_1, b_2, I_{\alpha,2}](f_1, f_2)\|_{L^q(\mu)} \leq C \|f_1\|_{L^{p_1}(\mu)} \|f_2\|_{L^{p_2}(\mu)},$$

where $\frac{1}{q} = \frac{1}{p_1} + \frac{1}{p_2} - \alpha$.

Remark 1.12. For $\|\mu\| < \infty$, by Lemma 2.1 in Section 2, Theorem 1.11 also holds if one assumes that $\int_X G(f_1, f_2)(x) d\mu(x) = 0$ with the operator G be replaced by $I_{\alpha,2}$, $[b_1, I_{\alpha,2}]$, $[b_2, I_{\alpha,2}]$ and $[b_1, b_2, I_{\alpha,2}]$.

This paper is organized as follows. Theorem 1.10 and Theorem 1.11 are proved in Section 2. In Section 3, some applications are stated. Throughout this paper, C always denotes a positive constant independent of the main parameters involved, but it may be different from line to line.

2. PROOF OF MAIN RESULTS

Proof of Theorem 1.10. Let $\alpha = \alpha_1 + \alpha_2$, $0 < \alpha_i < 1/p_i < 1$ for $i = 1, 2$. It is easy to check that

$$\prod_{j=1}^2 [\lambda(x, d(x, y_j))]^{1-\alpha_i} \leq \left[\sum_{j=1}^2 \lambda(x, d(x, y_j)) \right]^{2-\alpha},$$

thus

$$|I_{\alpha,2}(f_1, f_2)(x)| \leq \prod_{j=1}^2 I_{\alpha_i}(|f_i|)(x).$$

Let $1/q_i = 1/p_i - \alpha_i$ and $1/q_1 + 1/q_2 = 1/q$, $1 < q_i < \infty$. It follows from the Hölder's inequality and the boundedness of I_{α_i} , $i = 1, 2$ that

$$\begin{aligned} & \|I_{\alpha,2}(f_1, f_2)\|_{L^q(\mu)} \\ & \leq \left\| \prod_{j=1}^2 I_{\alpha_i}(|f_i|) \right\|_{L^q(\mu)} \\ & \leq \|I_{\alpha_1}(|f_1|)\|_{L^{q_1}(\mu)} \|I_{\alpha_2}(|f_2|)(x)\|_{L^{q_2}(\mu)} \\ & \leq \|f_1\|_{L^{p_1}(\mu)} \|f_2\|_{L^{p_2}(\mu)}. \end{aligned}$$

The proof of Theorem 1.10 is completed. \square

To prove Theorem 1.11, first we give some notations and lemmas.

Let $f \in L^1_{loc}(\mu)$ and $0 < \beta < 1$, the sharp maximal operator is

$$M^{\sharp,(\beta)} f(x) = \sup_{B \ni x} \frac{1}{\mu(6B)} \int_B |f(y) - m_{\tilde{B}}(f)| d\mu(y) + \sup_{(B,Q) \in \Delta_x} \frac{|m_B(f) - m_Q(f)|}{K_{B,Q}^{(\beta)}},$$

where $\Delta_x := \{(B, Q) : x \in B \subset Q \text{ and } B, Q \text{ are doubling balls}\}$.

The non centered doubling maximal operator is

$$Nf(x) = \sup_{B \ni x, B \text{ doubling}} \frac{1}{\mu(B)} \int_B |f(y)| d\mu(y).$$

By the Lebesgue differential theorem, for any $f \in L^1_{loc}(\mu)$, we have

$$|f(x)| \leq Nf(x)$$

for $\mu - a.e. x \in X$.

Set $\rho > 1$, $p \in (1, \infty)$ and $r \in (1, p)$, the non-centered maximal operator $M_{r,(\rho)}^{(\alpha)} f$ is defined by

$$M_{r,(\rho)}^{(\alpha)} f(x) = \sup_{B \ni x} \left\{ \frac{1}{[\mu(\rho B)]^{1-\alpha r}} \int_B |f(y)|^r d\mu(y) \right\}^{1/r}.$$

When $r = 1$, we simply write $M_{1,(\rho)}^{(0)} f(x)$ as $M_{(\rho)} f$. If $\rho \geq 5$, then the operator $M_{(\rho)} f$ is bounded on $L^p(\mu)$ for any $p > 1$, and $M_{r,(\rho)}^{(\alpha)}$ is bounded from $L^p(\mu)$ to $L^q(\mu)$ for $p \in (r, 1/\alpha)$ and $1/q = 1/p - \alpha$ (see [6]).

Lemma 2.1. [6] *For $\|\mu\| < \infty$, if $f \in L^1_{loc}(\mu)$, $\int_X f(x) d\mu(x) = 0$, $1 < p < \infty$, $0 < \delta < 1$, and $\inf(1, Nf) \in L^p(\mu)$, for $0 < \beta < 1$, then there exists a constant $C > 0$ such that*

$$\|N(f)\|_{L^p(\mu)} \leq C \|M^{\sharp,(\beta)}(f)\|_{L^p(\mu)}.$$

Lemma 2.2. [1, 25] *$1 \leq p < \infty$ and $1 < \rho < \infty$, then $b \in RBMO(\mu)$ if and only if for any ball $B \in X$,*

$$\left\{ \frac{1}{\mu(\rho B)} \int_B |b_B - m_{\tilde{B}}(b)|^p d\mu(X) \right\}^{1/p} \leq C \|b\|_*,$$

and for any two doubling ball $B \subset Q$,

$$|m_B(b) - m_Q(b)| \leq CK_{B,Q} \|b\|_*. \quad (2.1)$$

Lemma 2.3. [10]

$$|m_{\widetilde{6^j \frac{6}{5} B}}(b) - m_{\tilde{B}}(b)| \leq Cj \|b\|_*.$$

Lemma 2.4. *Suppose $0 < \alpha < 2$, $1 < p_1, p_2, q < \infty$, $1 < r < q$ and $b_1, b_2 \in RBMO(\mu)$. If I_β is bounded from L^r into L^s for all $r \in (1, 1/\beta)$ and $1/s = 1/r - \beta$ with $0 < \beta < 1$, then there exists a constant $C > 0$ such that for any $x \in X$, $f_1 \in L^{p_1}(\mu)$ and $f_2 \in L^{p_2}(\mu)$,*

$$\begin{aligned} & M^{\sharp,(\alpha/2)}[b_1, b_2, I_{\alpha,2}](f_1, f_2)(x) \\ & \leq C \{ \|b_1\|_* \|b_2\|_* M_{r,(6)}(I_{\alpha,2}(f_1, f_2))(x) + \|b_1\|_* M_{r,(6)}([b_2, I_{\alpha,2}](f_1, f_2))(x) \\ & + \|b_2\|_* M_{r,(6)}([b_1, I_{\alpha,2}](f_1, f_2))(x) + \|b_1\|_* \|b_2\|_* M_{p_1,(5)}^{(\alpha/2)} f_1(x) M_{p_2,(5)} f_2(x) \}, \end{aligned} \quad (2.2)$$

$$\begin{aligned} & M^{\sharp,(\alpha/2)}[b_1, I_{\alpha,2}](f_1, f_2)(x) \\ & \leq C \{ \|b_1\|_* M_{r,(6)}(I_{\alpha,2}(f_1, f_2))(x) + \|b_1\|_* M_{p_1,(5)}^{(\alpha/2)} f_1(x) M_{p_2,(5)} f_2(x) \}, \end{aligned} \quad (2.3)$$

and

$$\begin{aligned} & M^{\sharp, (\alpha/2)}[b_2, I_{\alpha, 2}](f_1, f_2)(x) \\ & \leq C \{ \|b_2\|_* M_{r, (6)}(I_{\alpha, 2}(f_1, f_2))(x) + \|b_2\|_* M_{p_1, (5)}^{(\alpha/2)} f_1(x) M_{p_2, (5)}^{(\alpha/2)} f_2(x) \}. \end{aligned} \quad (2.4)$$

Proof. As $L^\infty(\mu)$ with compact support is dense in $L^p(\mu)$ for $1 < p < \infty$, we only consider $f_1, f_2 \in L^\infty(\mu)$ with compact support. Also, by Corollary 3.11 in [4], without loss of generality, we assume $b_1, b_2 \in L^\infty(\mu)$.

As Theorem 9.1 in [25], in order to obtain (2.2), it suffices to show that

$$\begin{aligned} & \frac{1}{\mu(6B)} \int_B |[b_1, b_2, I_{\alpha, 2}](f_1, f_2)(z) - h_B| d\mu(z) \\ & \leq C \{ \|b_1\|_* \|b_2\|_* M_{r, (6)}(I_{\alpha, 2}(f_1, f_2))(x) + \|b_1\|_* M_{r, (6)}([b_2, I_{\alpha, 2}](f_1, f_2))(x) \\ & \quad + \|b_2\|_* M_{r, (6)}([b_1, I_{\alpha, 2}](f_1, f_2))(x) + C \|b_1\|_* \|b_2\|_* M_{p_1, (5)}^{(\alpha/2)} f_1(x) M_{p_2, (5)}^{(\alpha/2)} f_2(x) \}, \end{aligned} \quad (2.5)$$

holds for any $x \in B$, and

$$\begin{aligned} & |h_B - h_Q| \\ & \leq CK_{B, Q}^2 K_{B, Q}^{(\alpha/2)} \left[\|b_1\|_* \|b_2\|_* M_{r, (6)}(I_{\alpha, 2}(f_1, f_2))(x) \right. \\ & \quad + \|b_1\|_* \|b_2\|_* M_{p_1, (5)}^{(\alpha/2)} f_1(x) M_{p_2, (5)}^{(\alpha/2)} f_2(x) \\ & \quad \left. + \|b_1\|_* M_{r, (6)}([b_2, I_{\alpha, 2}](f_1, f_2))(x) + \|b_2\|_* M_{r, (6)}([b_1, I_{\alpha, 2}](f_1, f_2))(x) \right]. \end{aligned} \quad (2.6)$$

for any ball $B \subset Q$ with $x \in B$, where Q is a doubling ball. For any ball B , denote

$$h_B := m_B(I_{\alpha, 2}((b_1 - m_{\tilde{B}}(b_1))f_1 \chi_{X \setminus \frac{6}{5}B}, (b_2 - m_{\tilde{B}}(b_2))f_2 \chi_{X \setminus \frac{6}{5}B})),$$

and

$$h_Q := m_Q(I_{\alpha, 2}((b_1 - m_Q(b_1))f_1 \chi_{X \setminus \frac{6}{5}Q}, (b_2 - m_Q(b_2))f_2 \chi_{X \setminus \frac{6}{5}Q})).$$

Since

$$[b_1, b_2, I_{\alpha, 2}] = I_{\alpha, 2}((b_1 - b_1(z))f_1, (b_2 - b_2(z))f_2),$$

and

$$\begin{aligned} & I_{\alpha, 2}((b_1 - m_{\tilde{B}}(b_1))f_1, (b_2 - m_{\tilde{B}}(b_2))f_2) \\ & = I_{\alpha, 2}((b_1 - b_1(z) + b_1(z) - m_{\tilde{B}}(b_1))f_1, (b_2 - b_2(z) + b_2(z) - m_{\tilde{B}}(b_2))f_2) \\ & = (b_1(z) - m_{\tilde{B}}(b_1))(b_2(z) - m_{\tilde{B}}(b_2))I_{\alpha, 2}(f_1, f_2) \\ & \quad - (b_1(z) - m_{\tilde{B}}(b_1))I_{\alpha, 2}(f_1, (b_2 - b_2(z))f_2) \\ & \quad - (b_2(z) - m_{\tilde{B}}(b_2))I_{\alpha, 2}((b_1 - b_1(z))f_1, f_2) \\ & \quad + I_{\alpha, 2}((b_1 - b_1(z))f_1, (b_2 - b_2(z))f_2), \end{aligned} \quad (2.7)$$

it follows that

$$\begin{aligned}
& \left(\frac{1}{\mu(6B)} \int_B |[b_1, b_2, I_{\alpha,2}](f_1, f_2)(z) - h_B| d\mu(z) \right) \\
& \leq C \left(\frac{1}{\mu(6B)} \int_B |(b_1(z) - m_{\tilde{B}}(b_1))(b_2(z) - m_{\tilde{B}}(b_2))I_{\alpha,2}(f_1, f_2)(z)| d\mu(z) \right) \\
& + C \left(\frac{1}{\mu(6B)} \int_B |(b_1(z) - m_{\tilde{B}}(b_1))I_{\alpha,2}(f_1, (b_2 - b_2(z))f_2)(z)| d\mu(z) \right) \\
& + C \left(\frac{1}{\mu(6B)} \int_B |(b_2(z) - m_{\tilde{B}}(b_2))I_{\alpha,2}((b_1 - b_1(z))f_1, f_2)(z)| d\mu(z) \right) \\
& + C \left(\frac{1}{\mu(6B)} \int_B |I_{\alpha,2}((b_1 - m_{\tilde{B}}(b_1))f_1, (b_2 - m_{\tilde{B}}(b_2))f_2)(z) - h_B| d\mu(z) \right) \\
& =: E_1 + E_2 + E_3 + E_4. \tag{2.8}
\end{aligned}$$

For E_1 , let $1 < r_1, r_2$ such that $\frac{1}{r} + \frac{1}{r_1} + \frac{1}{r_2} = 1$. It follows from Hölder's inequality that

$$\begin{aligned}
& E_1 \\
& \leq C \left(\frac{1}{\mu(6B)} \int_B |b_1(z) - m_{\tilde{B}}b_1|^{r_1} d\mu(z) \right)^{1/r_1} \\
& \quad \times \left(\frac{1}{\mu(6B)} \int_B |b_2(z) - m_{\tilde{B}}b_2|^{r_2} d\mu(z) \right)^{1/r_2} \\
& \quad \times \left(\frac{1}{\mu(6B)} \int_B |I_{\alpha,2}(f_1, f_2)|^r d\mu(z) \right)^{1/r} \\
& \leq C \|b_1\|_* \|b_2\|_* M_{r,(6)}(I_{\alpha,2}(f_1, f_2))(x).
\end{aligned}$$

For E_2 , let $1 < s$ such that $\frac{1}{s} + \frac{1}{r} = 1$, by Hölder's inequality, one deduces

$$\begin{aligned}
& E_2 \\
& \leq C \left(\frac{1}{\mu(6B)} \int_B |b_1(z) - m_{\tilde{B}}b_1|^s d\mu(z) \right)^{1/s} \\
& \quad \times \left(\frac{1}{\mu(6B)} \int_B |[b_2, I_{\alpha,2}](f_1, f_2)|^r d\mu(z) \right)^{1/r} \\
& \leq C \|b_1\|_* M_{r,(6)}([b_2, I_{\alpha,2}](f_1, f_2))(x).
\end{aligned}$$

For E_3 , in a similar way we can obtain

$$E_3 \leq C \|b_2\|_* M_{r,(6)}([b_1, I_{\alpha,2}](f_1, f_2))(x).$$

For E_4 , let $f_k^1 = f_k \chi_{\frac{6}{5}B}$ and $f_k^2 = f_k - f_k^1$ for $k = 1, 2$. Then

$$\begin{aligned}
& E_4 \\
& \leq C \left(\frac{1}{\mu(6B)} \int_B |I_{\alpha,2}((b_1 - m_{\tilde{B}}b_1)f_1^1(z), (b_2 - m_{\tilde{B}}b_2)f_2^1(z))| d\mu(z) \right) \\
& + C \left(\frac{1}{\mu(6B)} \int_B |I_{\alpha,2}((b_1 - m_{\tilde{B}}b_1)f_1^1(z), (b_2 - m_{\tilde{B}}b_2)f_2^2(z))| d\mu(z) \right) \\
& + C \left(\frac{1}{\mu(6B)} \int_B |I_{\alpha,2}((b_1 - m_{\tilde{B}}b_1)f_1^2(z), (b_2 - m_{\tilde{B}}b_2)f_2^1(z))| d\mu(z) \right) \\
& + C \left(\frac{1}{\mu(6B)} \int_B |I_{\alpha,2}((b_1 - m_{\tilde{B}}b_1)f_1^2(z), (b_2 - m_{\tilde{B}}b_2)f_2^2(z)) - h_B| d\mu(z) \right) \\
& =: E_{41} + E_{42} + E_{43} + E_{44}.
\end{aligned}$$

For $1 < p_i < \infty$, $i = 1, 2$, set $s_1 = \sqrt{p_1}$, $s_2 = \sqrt{p_2}$, $\frac{1}{v} = \frac{1}{s_1} + \frac{1}{s_2} - \alpha$, $\frac{1}{s_1} = \frac{1}{p_1} + \frac{1}{v_1}$ and $\frac{1}{s_2} = \frac{1}{p_2} + \frac{1}{v_2}$. It follows from Hölder's inequality and Theorem 1.10 that

$$\begin{aligned}
& E_{41} \\
& \leq C \frac{\mu(B)^{1-1/v}}{\mu(6B)} \|I_{\alpha,2}((b_1 - m_{\tilde{B}}b_1)f_1^1, (b_2 - m_{\tilde{B}}b_2)f_2^1)\|_{L^v(\mu)} \\
& \leq C \frac{1}{\mu(6B)^{1/v}} \|(b_1 - m_{\tilde{B}}b_1)f_1^1\|_{L^{s_1}(\mu)} \|(b_2 - m_{\tilde{B}}b_2)f_2^1\|_{L^{s_2}(\mu)} \\
& \leq C \frac{1}{\mu(6B)^{1/v}} \left(\int_{\frac{6}{5}B} |(b_1 - m_{\tilde{B}}b_1)|^{v_1} d\mu(z) \right)^{1/v_1} \left(\int_{\frac{6}{5}B} |f_1(z)|^{p_1} d\mu(z) \right)^{1/p_1} \\
& \quad \times \left(\int_{\frac{6}{5}B} |(b_2 - m_{\tilde{B}}b_2)|^{v_2} d\mu(z) \right)^{1/v_2} \left(\int_{\frac{6}{5}B} |f_2(z)|^{p_2} d\mu(z) \right)^{1/p_2} \\
& \leq C \prod_{i=1}^2 \left(\frac{\int_{\frac{6}{5}B} |b_i - m_{\tilde{B}}b_i|^{v_i} d\mu(z)}{\mu(6B)} \right)^{1/v_i} \left(\frac{\int_{\frac{6}{5}B} |f_i(z)|^{p_i} d\mu(z)}{\mu(6B)^{1-\alpha p_i/2}} \right)^{1/p_i} \\
& \leq C \|b_1\|_* \|b_2\|_* M_{p_1, (5)}^{(\alpha/2)} f_1(x) M_{p_2, (5)}^{(\alpha/2)} f_2(x).
\end{aligned}$$

For E_{42} , using (i) of Definition 1.5, Lemma 2.2, Lemma 2.3, Hölder's inequality and the condition of $\lambda(x, ar) \geq a^m \lambda(x, r)$, we have

$$\begin{aligned}
E_{42} &\leq C \frac{1}{\mu(6B)} \int_B \int_X \int_X \frac{|b_1(y_1) - m_{\bar{B}} b_1| |f_1^1(y_1)|}{[\lambda(z, d(z, y_1)) + \lambda(z, d(z, y_2))]^{2-\alpha}} \\
&\quad \times |b_2(y_2) - m_{\bar{B}} b_2| |f_2^2(y_2)| d\mu(y_1) d\mu(y_2) d\mu(z) \\
&\leq C \frac{1}{\mu(6B)} \int_B \int_{\frac{6}{5}B} |b_1(y_1) - m_{\bar{B}} b_1| |f_1(y_1)| d\mu(y_1) \\
&\quad \times \int_{X \setminus \frac{6}{5}B} \frac{|b_2(y_2) - m_{\bar{B}} b_2| |f_2(y_2)| d\mu(y_2)}{[\lambda(z, d(z, y_2))]^{2-\alpha}} d\mu(z) \\
&\leq C \left(\frac{1}{\mu(6B)} \int_{\frac{6}{5}B} |b_1(y_1) - m_{\bar{B}} b_1|^{p'_1} d\mu(y_1) \right)^{1/p'_1} \\
&\quad \times \left(\frac{1}{\mu(6B)^{1-\alpha p_1/2}} \int_{\frac{6}{5}B} |f_1(y_1)|^{p_1} d\mu(y_1) \right)^{1/p_1} \\
&\quad \times \mu(6B)^{-\alpha/2} \mu(B) \sum_{k=1}^{\infty} \int_{6^k \frac{6}{5}B \setminus 6^{k-1} \frac{6}{5}B} \frac{|b_2(y_2) - m_{\bar{B}} b_2| |f_2(y_2)|}{[\lambda(x, 6^{k-1} \frac{6}{5} r_B)]^{2-\alpha}} d\mu(y_2) \\
&\leq C \|b_1\|_* M_{p_1, (5)}^{(\alpha/2)} f_1(x) \sum_{k=1}^{\infty} 6^{-km(1-\alpha/2)} \left[\frac{\mu(B)}{\mu(\frac{6}{5}B)} \right]^{1-\alpha/2} \left[\frac{\mu(\frac{6}{5}B)}{\lambda(x, \frac{6}{5} r_B)} \right]^{1-\alpha/2} \\
&\quad \times \frac{1}{[\lambda(x, 5 \times 6^k \frac{6}{5} r_B)]^{1-\alpha/2}} \int_{6^k \frac{6}{5}B} |b_2(y_2) - m_{\bar{B}} b_2| |f_2(y_2)| d\mu(y_2) \\
&\leq C \|b_1\|_* M_{p_1, (5)}^{(\alpha/2)} f_1(x) \sum_{k=1}^{\infty} 6^{-km(1-\alpha/2)} \frac{1}{[\mu(5 \times 6^k \frac{6}{5} B)]^{1-\alpha/2}} \\
&\quad \times \int_{6^k \frac{6}{5}B} |b_2(y_2) - m_{\widetilde{6^k \frac{6}{5}B}}(b_2) + m_{\widetilde{6^k \frac{6}{5}B}}(b_2) - m_{\bar{B}} b_2| |f_2(y_2)| d\mu(y_2) \\
&\leq C \|b_1\|_* M_{p_1, (5)}^{(\alpha/2)} f_1(x) \sum_{k=1}^{\infty} 6^{-km(1-\alpha/2)} \\
&\quad \times \left[\left(\frac{1}{\mu(5 \times 6^k \frac{6}{5} B)} \int_{6^k \frac{6}{5}B} |b_2(y_2) - m_{\widetilde{6^k \frac{6}{5}B}}(b_2)|^{p'_2} d\mu(y_2) \right)^{1/p'_2} \right. \\
&\quad \times \left(\frac{1}{\mu(5 \times 6^k \frac{6}{5} B)^{1-\alpha p_2/2}} \int_{6^k \frac{6}{5}B} |f_2(y_2)|^{p_2} d\mu(y_2) \right)^{1/p_2} \\
&\quad \left. + Ck \|b_2\|_* l \left(\frac{1}{\mu(5 \times 6^k \frac{6}{5} B)^{1-\alpha p_2/2}} \int_{6^k \frac{6}{5}B} |f_2(y_2)|^{p_2} d\mu(y_2) \right)^{1/p_2} \right. \\
&\quad \times \left. \left(\frac{1}{\mu(5 \times 6^k \frac{6}{5} B)} \int_{6^k \frac{6}{5}B} d\mu(y_2) \right)^{1/p'_2} \right] \\
&\leq C \|b_1\|_* \|b_2\|_* M_{p_1, (5)}^{(\alpha/2)} f_1(x) M_{p_2, (5)}^{(\alpha/2)} f_2(x).
\end{aligned}$$

Similarly, we get

$$E_{43} \leq C \|b_1\|_* \|b_2\|_* M_{p_1, (5)}^{(\alpha/2)} f_1(x) M_{p_2, (5)}^{(\alpha/2)} f_2(x).$$

For E_{44} , by (ii) of Definition 1.5, Lemma 2.2, Lemma 2.3, Hölder's inequality and the properties of λ , we obtain

$$\begin{aligned} & |I_{\alpha, 2}((b_1 - m_{\tilde{B}} b_1) f_2^2, (b_2 - m_{\tilde{B}} b_2) f_2^2)(z) \\ & \quad - I_{\alpha, 2}((b_1 - m_{\tilde{B}} b_1) f_2^2, (b_2 - m_{\tilde{B}} b_2) f_2^2)(z_0)| \\ & \leq C \int_{X \setminus \frac{6}{5}B} \int_{X \setminus \frac{6}{5}B} |K(z, y_1, y_2) - K(z_0, y_1, y_2)| \\ & \quad \prod_{i=1}^2 |(b_i(y_i) - m_{\tilde{B}} b_i) f_i(y_i)| d\mu(y_i) \\ & \leq C \int_{X \setminus \frac{6}{5}B} \int_{X \setminus \frac{6}{5}B} \frac{d(z, z_0)^\delta \prod_{i=1}^2 |(b_i(y_i) - m_{\tilde{B}} b_i) f_i(y_i)| d\mu(y_i)}{(d(z, y_1) + d(z, y_2))^\delta [\sum_{j=1}^2 \lambda(x, d(x, y_j))]^{2-\alpha}} \\ & \leq C \prod_{i=1}^2 \int_{X \setminus \frac{6}{5}B} \frac{d(z, z_0)^{\delta_i} |b_i(y_i) - m_{\tilde{B}} b_i| |f_i(y_i)| d\mu(y_i)}{d(z, y_i)^{\delta_i} [\lambda(z, d(z, y_i))]^{1-\alpha/2}} \\ & \leq C \prod_{i=1}^2 \sum_{k=1}^{\infty} \int_{6^k \frac{6}{5}B \setminus 6^{k-1} \frac{6}{5}B} 6^{-k\delta_i} \left[\frac{\mu(5 \times 6^k \frac{6}{5}B)}{\lambda(z, 5 \times 6^k \frac{6}{5}r_B)} \right]^{1-\alpha/2} \frac{1}{[\mu(5 \times 6^k \frac{6}{5}B)]^{1-\alpha/2}} \\ & \quad \times |b_i(y_i) - m_{\tilde{B}} b_i| |f_i| d\mu(y_i) \\ & \leq C \prod_{i=1}^2 \sum_{k=1}^{\infty} 6^{-k\delta_i} \left(\frac{1}{[\mu(5 \times 6^k \frac{6}{5}B)]^{1-\alpha p_i/2}} \int_{6^k \frac{6}{5}B} |b_i(y_i) - m_{\tilde{B}} b_i|^{p'_i} d\mu(y_i) \right)^{1/p'_i} \\ & \quad \times \left(\frac{1}{\mu(5 \times 6^k \frac{6}{5}B)} \int_{6^k \frac{6}{5}B} |f_i|^{p_i} \right)^{1/p_i} \\ & \leq C \prod_{i=1}^2 \sum_{k=1}^{\infty} 6^{-k\delta_i} M_{p_i, (6)} f_i(x) \left(\frac{1}{[\mu(5 \times 6^k \frac{6}{5}B)]^{1-\alpha p_i/2}} \int_{6^k \frac{6}{5}B} |b_i(y_i) - m_{\widetilde{6^k \frac{6}{5}B}}|^{p'_i} \right. \\ & \quad \left. + m_{\widetilde{6^k \frac{6}{5}B}} - m_{\tilde{B}} b_i|^{p'_i} d\mu(y_i) \right)^{1/p'_i} \\ & \leq C \prod_{i=1}^2 \sum_{k=1}^{\infty} 6^{-k\delta_i} k \|b_i\|_* M_{p_i, (6)} f_i(x) \\ & \leq C \|b_1\|_* \|b_2\|_* M_{p_1, (6)}^{(\alpha/2)} f_1(x) M_{p_2, (6)}^{(\alpha/2)} f_2(x). \end{aligned}$$

where $\delta_1, \delta_2 > 0$ and $\delta_1 + \delta_2 = \delta$.

Taking the mean over $z_0 \in B$, it deduces

$$E_{44} \leq C \|b_1\|_* \|b_2\|_* M_{p_1, (6)}^{(\alpha/2)} f_1(x) M_{p_2, (6)}^{(\alpha/2)} f_2(x). \quad (2.9)$$

So (2.5) can be obtain from (2.8) to (2.9).

Next we prove (2.6). Consider two balls $B \subset Q$ with $x \in B$, where B is an arbitrary ball and Q is a doubling ball. Let $N = N_{B,Q} + 1$, then we yield

$$\begin{aligned}
& \left| m_B[I_{\alpha,2}((b_1 - m_{\tilde{B}}b_1)f_1^2, (b_2 - m_{\tilde{B}}b_2)f_2^2)] \right. \\
& \quad \left. - m_Q[I_{\alpha,2}((b_1 - m_Qb_1)f_1^2, (b_2 - m_Qb_2)f_2^2)] \right| \\
& \leq |m_B[I_{\alpha,2}((b_1 - m_{\tilde{B}}b_1)f_1\chi_{X \setminus 6^NB}, (b_2 - m_{\tilde{B}}b_2)f_2\chi_{X \setminus 6^NB})] \\
& \quad - m_Q[I_{\alpha,2}((b_1 - m_{\tilde{B}}b_1)f_1\chi_{X \setminus 6^NB}, (b_2 - m_{\tilde{B}}b_2)f_2\chi_{X \setminus 6^NB})]| \\
& \quad + |m_Q[I_{\alpha,2}((b_1 - m_Qb_1)f_1\chi_{X \setminus 6^NB}, (b_2 - m_Qb_2)f_2\chi_{X \setminus 6^NB})] \\
& \quad - m_Q[I_{\alpha,2}((b_1 - m_{\tilde{B}}b_1)f_1\chi_{X \setminus 6^NB}, (b_2 - m_{\tilde{B}}b_2)f_2\chi_{X \setminus 6^NB})]| \\
& \quad + |m_B[I_{\alpha,2}((b_1 - m_{\tilde{B}}b_1)f_1\chi_{6^NB \setminus \frac{6}{5}B}, (b_2 - m_{\tilde{B}}b_2)f_2\chi_{X \setminus \frac{6}{5}B})] \\
& \quad + |m_B[I_{\alpha,2}((b_1 - m_{\tilde{B}}b_1)f_1\chi_{X \setminus \frac{6}{5}B}, (b_2 - m_{\tilde{B}}b_2)f_2\chi_{6^NB \setminus \frac{6}{5}B})] \\
& \quad + |m_Q[I_{\alpha,2}((b_1 - m_Qb_1)f_1\chi_{6^NB \setminus \frac{6}{5}Q}, (b_2 - m_Qb_2)f_2\chi_{X \setminus 6^NB})] \\
& \quad + |m_Q[I_{\alpha,2}((b_1 - m_Qb_1)f_1\chi_{X \setminus \frac{6}{5}Q}, (b_2 - m_Qb_2)f_2\chi_{6^NB \setminus \frac{6}{5}Q})]| \\
& =: F_1 + F_2 + F_3 + F_4 + F_5 + F_6.
\end{aligned} \tag{2.10}$$

Using the method to estimate E_{44} , we get

$$F_1 \leq C \|b_1\|_* \|b_2\|_* M_{p_1, (6)}^{(\alpha/2)} f_1(x) M_{p_2, (6)}^{(\alpha/2)} f_2(x).$$

Let us estimate F_2 . At first, we calculate

$$\begin{aligned}
& I_{\alpha,2}((b_1 - m_Qb_1)f_1\chi_{X \setminus 6^NB}, (b_2 - m_Qb_2)f_2\chi_{X \setminus 6^NB})(z) \\
& \quad - I_{\alpha,2}((b_1 - m_{\tilde{B}}b_1)f_1\chi_{X \setminus 6^NB}, (b_2 - m_{\tilde{B}}b_2)f_2\chi_{X \setminus 6^NB})(z) \\
& = (m_Qb_2 - m_{\tilde{B}}b_2)I_{\alpha,2}((b_1 - m_Qb_1)f_1\chi_{X \setminus 6^NB}, f_2\chi_{X \setminus 6^NB})(z) \\
& \quad + (m_Qb_1 - m_{\tilde{B}}b_1)I_{\alpha,2}(f_1\chi_{X \setminus 6^NB}, (b_2 - m_Qb_2)f_2\chi_{X \setminus 6^NB})(z) \\
& \quad + (m_Qb_1 - m_{\tilde{B}}b_1)(m_Qb_2 - m_{\tilde{B}}b_2)I_{\alpha,2}(f_1\chi_{X \setminus 6^NB}, f_2\chi_{X \setminus 6^NB})(z).
\end{aligned}$$

Hence

$$\begin{aligned}
& F_2 \\
& \leq |(m_Qb_2 - m_{\tilde{B}}b_2) \frac{1}{\mu(Q)} \int_Q I_{\alpha,2}((b_1 - m_Qb_1)f_1\chi_{X \setminus 6^NB}, f_2\chi_{X \setminus 6^NB})(z) d\mu(z)| \\
& \quad + |(m_Qb_1 - m_{\tilde{B}}b_1) \frac{1}{\mu(Q)} \int_Q I_{\alpha,2}(f_1\chi_{X \setminus 6^NB}, (b_2 - m_Qb_2)f_2\chi_{X \setminus 6^NB})(z) d\mu(z)| \\
& \quad + |(m_Qb_1 - m_{\tilde{B}}b_1)(m_Qb_2 - m_{\tilde{B}}b_2) \frac{1}{\mu(Q)} \int_Q I_{\alpha,2}(f_1\chi_{X \setminus 6^NB}, f_2\chi_{X \setminus 6^NB})(z)| \\
& =: F_{21} + F_{22} + F_{23}.
\end{aligned}$$

To estimate F_{21} , we write

$$\begin{aligned}
& I_{\alpha,2}((b_1 - m_Q b_1) f_1 \chi_{X \setminus 6^N Q}, f_2 \chi_{X \setminus 6^N Q})(z) \\
&= I_{\alpha,2}((b_1 - m_Q b_1) f_1, f_2)(z) - T((b_1 - m_Q b_1) f_1 \chi_{6^N B \setminus \frac{6}{5} Q}, f_2 \chi_{\frac{6}{5} Q})(z) \\
&\quad - I_{\alpha,2}((b_1 - m_Q b_1) f_1 \chi_{\frac{6}{5} Q}, f_2 \chi_{6^N B \setminus \frac{6}{5} Q})(z) \\
&\quad + I_{\alpha,2}((b_1 - m_Q b_1) f_1 \chi_{6^N B \setminus \frac{6}{5} Q}, f_2 \chi_{6^N B \setminus \frac{6}{5} Q})(z) \\
&\quad - I_{\alpha,2}((b_1 - m_Q b_1) f_1 \chi_{X \setminus \frac{6}{5} Q}, f_2 \chi_{6^N B})(z) \\
&\quad - I_{\alpha,2}((b_1 - m_Q b_1) f_1 \chi_{6^N B}, f_2 \chi_{X \setminus \frac{6}{5} Q})(z) \\
&\quad + I_{\alpha,2}((b_1 - m_Q b_1) f_1 \chi_{6^N B \setminus \frac{6}{5} Q}, f_2 \chi_{6^N B \setminus \frac{6}{5} Q})(z) \\
&=: H_1(z) + H_2(z) + H_3(z) + H_4(z) + H_5(z) + H_6(z) + H_7(z).
\end{aligned}$$

Let us first estimate $H_1(z)$. It is easy to see that

$$\frac{1}{\mu(Q)} \int_Q |I_{\alpha,2}(b_1 - b_1(z) f_1, f_2)(z)| d\mu(z) \leq C M_{r,(6)}([b_1, I_{\alpha,2}] f_1, f_2)(x).$$

By Hölder's inequality, we have

$$\frac{1}{\mu(Q)} \int_Q |(b_1(z) - m_Q(b_1)) I_{\alpha,2}(f_1, f_2)(z)| d\mu(z) \leq C \|b_1\|_* M_{r,(6)}(I_{\alpha,2}(f_1, f_2))(x).$$

Then we obtain

$$\begin{aligned}
& |m_Q(H_1)| \\
&\leq |m_Q(I_{\alpha,2}(b_1 - b_1(z) f_1, f_2))| + |m_Q((b_1(z) - m_Q(b_1)) I_{\alpha,2}(f_1, f_2))| \\
&\leq C M_{r,(6)}([b_1, I_{\alpha,2}] f_1, f_2)(x) + \|b_1\|_* M_{r,(6)}(I_{\alpha,2}(f_1, f_2))(x).
\end{aligned}$$

For $H_2(z)$, $s_1 = \sqrt{p_1}$, $s_2 = p_2$, $\frac{1}{v} = \frac{1}{s_1} + \frac{1}{s_2} - \alpha$ and $\frac{1}{s_1} = \frac{1}{p_1} + \frac{1}{v_1}$. Using the fact that Q is a doubling balls, Lemma 3.1 and Hölder's inequality, we yield

$$\begin{aligned}
& |m_Q(H_2)| \\
&\leq C \frac{\mu(Q)^{1-1/v}}{\mu(6Q)} \|I_{\alpha,2}((b_1 - m_Q b_1) f_1 \chi_{6^N B \setminus \frac{6}{5} Q}, f_2 \chi_{\frac{6}{5} Q})\|_{L^v(\mu)} \\
&\leq C \mu(6Q)^{-1/v} \|(b_1 - m_Q b_1) f_1 \chi_{6^N B \setminus \frac{6}{5} Q}\|_{L^{s_1}(\mu)} \|f_2 \chi_{\frac{6}{5} Q}\|_{L^{s_2}(\mu)} \\
&\leq C \frac{1}{\mu(6Q)^{1/v}} \left(\int_{\frac{6}{5} Q} |b_1 - m_Q b_1|^{v_1} d\mu(z) \right)^{1/v_1} \left(\int_{\frac{6}{5} Q} |f_1(z)|^{p_1} d\mu(z) \right)^{1/p_1} \\
&\quad \times \left(\int_{\frac{6}{5} Q} |f_2(z)|^{p_2} d\mu(z) \right)^{1/p_2} \\
&\leq C \left(\frac{1}{\mu(6Q)} \int_{\frac{6}{5} Q} |b_1 - m_Q b_1|^{v_1} d\mu(z) \right)^{1/v_1} \\
&\quad \times \prod_{i=1}^2 \left(\frac{1}{\mu(6Q)^{1-\alpha p_i/2}} \int_{\frac{6}{5} Q} |f_i(z)|^{p_i} d\mu(z) \right)^{1/p_i} \\
&\leq C \|b_1\|_* M_{p_1,(5)}^{(\alpha/2)} f_1(x) M_{p_2,(5)}^{(\alpha/2)} f_2(x).
\end{aligned}$$

We can also obtain

$$|m_Q(H_3)| + |m_Q(H_4)| \leq C \|b_1\|_* M_{p_1, (5)}^{(\alpha/2)} f_1(x) M_{p_2, (5)}^{(\alpha/2)} f_2(x).$$

For H_5 , since $z \in Q$, by (i) of Definition 1.5, Lemma 2.2, Lemma 2.3, Hölder's inequality and the properties of λ and Q is a doubling ball, we deduce

$$\begin{aligned}
& |H_5(z)| \\
& \leq C \int_{6^N B} \int_{X \setminus \frac{6}{5}Q} \frac{|b_1(y_1) - m_Q b_1| |f_1(y_1)| |f_2(y_2)| d\mu(y_1) d\mu(y_2)}{[\sum_{j=1}^2 \lambda(x, d(x, y_j))]^{2-\alpha}} \\
& \leq C \int_{6^N B} |f_2(y_2)| d\mu(y_2) \sum_{k=1}^{\infty} \int_{6^k \frac{6}{5}Q} \frac{|b_1(y_1) - m_Q b_1| |f_1(y_1)|}{(\lambda(z, 5 \times 6^k \frac{6}{5} r_Q))^{2-\alpha}} d\mu(y_1) \\
& \leq C \int_{6^N B} |f_2(y_2)| d\mu(y_2) \sum_{k=1}^{\infty} 6^{-km(1-\alpha/2)} \\
& \quad \times \int_{6^k \frac{6}{5}Q} \frac{1}{[\lambda(z, 5 \times \frac{6}{5} r_Q)]^{1-\alpha/2}} \frac{|b_1(y_1) - m_Q b_1| |f_1(y_1)| d\mu(y_1)}{[\lambda(z, 5 \times 6^k \frac{6}{5} r_Q)]^{1-\alpha/2}} \\
& \leq C \frac{1}{[\lambda(z, 6r_Q)]^{1-\alpha/2}} \int_{6^N B} |f_2(y_2)| d\mu(y_2) \sum_{k=1}^{\infty} 6^{-km(1-\alpha/2)} \\
& \quad \times \frac{1}{[\lambda(z, 5 \times 6^k \frac{6}{5} r_Q)]^{1-\alpha/2}} \times \left[\int_{6^k \frac{6}{5}Q} |b_1(y_1) - m_{6^k \frac{6}{5}Q}(b_1)| |f_1(y_1)| d\mu(y_1) \right. \\
& \quad \left. + \int_{6^k \frac{6}{5}Q} |m_{6^k \frac{6}{5}Q}(b_1) - m_Q b_1| |f_1(y_1)| d\mu(y_1) \right] \\
& \leq C \frac{1}{[\lambda(z, 6r_Q)]^{1-\alpha/2}} \int_{6^N B} |f_2(y_2)| d\mu(y_2) \sum_{k=1}^{\infty} 6^{-km(1-\alpha/2)} \\
& \quad \times \left[\left(\frac{1}{[\lambda(z, 5 \times 6^k \frac{6}{5} r_Q)]^{1-\alpha/2}} \int_{6^k \frac{6}{5}Q} |b_1(y_1) - m_{6^k \frac{6}{5}Q}(b_1)|^{p'_1} d\mu(y_1) \right)^{1/p'_1} \right. \\
& \quad \times \left(\frac{1}{[\lambda(z, 6^{k+1} \frac{6}{5} r_Q)]^{1-\alpha/2}} \int_{6^k \frac{6}{5}Q} |f_1(y_1)|^{p_1} d\mu(y_1) \right)^{1/p_1} \\
& \quad \left. + k \|b_1\|_* \frac{1}{[\lambda(z, 5 \times 6^k \frac{6}{5} r_Q)]^{1-\alpha/2}} \int_{6^k \frac{6}{5}Q} |f_1(y_1)| d\mu(y_1) \right] \\
& \leq C \|b_1\|_* M_{p_1, (5)}^{(\alpha/2)} f_1(x) \sum_{k=1}^N \frac{1}{[\lambda(z, 6r_Q)]^{1-\alpha/2}} \int_{6^k B} |f_2(y_2)| d\mu(y_2) \\
& \leq C \|b_1\|_* M_{p_1, (5)}^{(\alpha/2)} f_1(x) \sum_{k=1}^N \left[\frac{\mu(5 \times 6^k B)}{\lambda(z, 5 \times 6^k r_B)} \right]^{1-\alpha/2} \left[\frac{\lambda(z, 5 \times 6^k r_B)}{\lambda(z, 6r_Q)} \right]^{1-\alpha/2} \\
& \quad \frac{1}{[\mu(5 \times 6^k B)]^{1-\alpha/2}} \int_{6^k B} |f_2(y_2)| d\mu(y_2) \\
& \leq C K_{B, Q}^{(\alpha/2)} \|b_1\|_* M_{p_1, (5)}^{(\alpha/2)} f_1(x) M_{p_2, (5)}^{(\alpha/2)} f_2(x).
\end{aligned}$$

Then

$$|m_Q(H_5)| \leq CK_{B,Q}^{(\alpha/2)} \|b_1\|_* M_{p_1,(5)}^{(\alpha/2)} f_1(x) M_{p_2,(5)}^{(\alpha/2)} f_2(x).$$

In the similar way to estimate $m_Q(H_5)$, it follows that

$$|m_Q(H_6)| + |m_Q(H_7)| \leq CK_{B,Q}^{(\alpha/2)} \|b_1\|_* M_{p_1,(5)}^{(\alpha/2)} f_1(x) M_{p_2,(5)}^{(\alpha/2)} f_2(x).$$

From (2.1) in Lemma 2.2, we deduce

$$\begin{aligned} & F_{21} \\ & \leq C \left[\|b_1\|_* \|b_2\|_* M_{r,(6)}(I_{\alpha,2}(f_1, f_2))(x) + \|b_1\|_* M_{r,(6)}([b_2, I_{\alpha,2}](f_1, f_2))(x) \right. \\ & \quad \left. + \|b_2\|_* M_{r,(6)}([b_1, I_{\alpha,2}](f_1, f_2))(x) + \|b_1\|_* \|b_2\|_* M_{p_1,(5)}^{(\alpha/2)} f_1(x) M_{p_2,(5)}^{(\alpha/2)} f_2(x) \right]. \end{aligned}$$

F_{22} and F_{23} also have similar estimate of F_{21} , therefore,

$$\begin{aligned} & F_2 \\ & \leq C \left[\|b_1\|_* \|b_2\|_* M_{r,(6)}(I_{\alpha,2}(f_1, f_2))(x) + \|b_1\|_* M_{r,(6)}([b_2, I_{\alpha,2}](f_1, f_2))(x) \right. \\ & \quad \left. + \|b_2\|_* M_{r,(6)}([b_1, I_{\alpha,2}](f_1, f_2))(x) + \|b_1\|_* \|b_2\|_* M_{p_1,(5)}^{(\alpha/2)} f_1(x) M_{p_2,(5)}^{(\alpha/2)} f_2(x) \right]. \end{aligned}$$

From F_3 to F_6 , using the similar method to estimate I_4 , we conclude

$$F_3 + F_4 + F_5 + F_6 \leq C \|b_1\|_* \|b_2\|_* M_{p_1,(5)}^{(\alpha/2)} f_1(x) M_{p_2,(5)}^{(\alpha/2)} f_2(x). \quad (2.11)$$

Thus (2.6) holds by from (2.10) to (2.11) and hence (2.2) is proved. With the same method to prove (2.2), we can obtain that (2.3) and (2.4) are also hold. Here we omit the details. Thus Lemma 2.4 is proved. \square

Proof of Theorem 1.11. Let $1 < p_1, p_2, q < \infty$, $\frac{1}{q} = \frac{1}{p_1} + \frac{1}{p_2}$, $1 < r < q$, $f_1 \in L^{p_1}(\mu)$, $f_2 \in L^{p_2}(\mu)$, $b_1 \in RBMO(\mu)$ and $b_2 \in RBMO(\mu)$. By $|f(x)| \leq Nf(x)$, Lemma 2.1-2.4, Theorem 1.10, Hölder's inequality and the boundedness of $M_{r,(\rho)}^{(\alpha/2)}$

and $M_{r,(\rho)}$ for $0 < \alpha < 2$, $5 \leq \rho$ and $r < q$, we obtain

$$\begin{aligned}
& \| [b_1, b_2, I_{\alpha,2}](f_1, f_2) \|_{L^q(\mu)} \\
& \leq \| N([b_1, b_2, I_{\alpha,2}](f_1, f_2)) \|_{L^q(\mu)} \\
& \leq C \| M^{\sharp,(\alpha/2)}([b_1, b_2, I_{\alpha,2}](f_1, f_2)) \|_{L^q(\mu)} \\
& \leq C \| b_1 \|_* \| b_2 \|_* \| M_{r,(6)}(I_{\alpha,2}(f_1, f_2)) \|_{L^q(\mu)} \\
& \quad + C \| b_1 \|_* \| M_{r,(6)}([b_2, I_{\alpha,2}](f_1, f_2)) \|_{L^q(\mu)} \\
& \quad + C \| b_2 \|_* \| M_{r,(6)}([b_1, I_{\alpha,2}](f_1, f_2)) \|_{L^q(\mu)} \\
& \quad + C \| b_1 \|_* \| b_2 \|_* \| M_{p_1,(5)}^{(\alpha/2)} f_1(x) M_{p_2,(5)}^{(\alpha/2)} f_2(x) \|_{L^q(\mu)} \\
& \leq C \| b_1 \|_* \| b_2 \|_* \| f_1(x) \|_{L^{p_1}(\mu)} \| f_2(x) \|_{L^{p_2}(\mu)} \\
& \quad + C \| b_1 \|_* \| ([b_2, I_{\alpha,2}](f_1, f_2)) \|_{L^q(\mu)} \\
& \quad + C \| b_2 \|_* \| ([b_1, I_{\alpha,2}](f_1, f_2)) \|_{L^q(\mu)} \\
& \leq C \| b_1 \|_* \| b_2 \|_* \| f_1(x) \|_{L^{p_1}(\mu)} \| f_2(x) \|_{L^{p_2}(\mu)} \\
& \quad + C \| b_1 \|_* \| M^{\sharp,(\alpha/2)}([b_2, I_{\alpha,2}](f_1, f_2)) \|_{L^q(\mu)} \\
& \quad + C \| b_2 \|_* \| M^{\sharp,(\alpha/2)}([b_1, I_{\alpha,2}](f_1, f_2)) \|_{L^q(\mu)} \\
& \leq \| b_1 \|_* \| b_2 \|_* \| f_1(x) \|_{L^{p_1}(\mu)} \| f_2(x) \|_{L^{p_2}(\mu)} \\
& \quad + C \| b_1 \|_* \| M_{r,(6)}(I_{\alpha,2}(f_1, f_2))(x) \|_{L^q(\mu)} \\
& \quad + C \| b_1 \|_* \| M_{p_1,(5)}^{(\alpha/2)} f_1(x) M_{p_2,(5)}^{(\alpha/2)} f_2(x) \|_{L^q(\mu)} \\
& \quad + C \| b_2 \|_* \| M_{r,(6)}(I_{\alpha,2}(f_1, f_2))(x) \|_{L^q(\mu)} \\
& \quad + C \| b_2 \|_* \| M_{p_1,(5)}^{(\alpha/2)} f_1(x) M_{p_2,(5)}^{(\alpha/2)} f_2(x) \|_{L^q(\mu)} \\
& \leq C \| b_1 \|_* \| b_2 \|_* \| f_1(x) \|_{L^{p_1}(\mu)} \| f_2(x) \|_{L^{p_2}(\mu)}.
\end{aligned}$$

Thus the proof of Theorem 1.11 is finished. \square

3. APPLICATIONS

In this section, we apply Theorems 1.10 and Theorem 1.11 to study a specific fractional integral operator.

Lemma 3.1. [6] *Assume that $\text{diam}(X) = \infty$. Let $\alpha \in (0, 1)$, $p \in (1, 1/\alpha)$ and $1/q = 1/p - \alpha$. If λ satisfies the ϵ -weak reverse doubling condition for some $\epsilon \in (0, \min\{\alpha, 1 - \alpha, 1/q\})$, then*

$$T_\alpha f(x) := \int_X \frac{f(y)}{[\lambda(y, d(x, y))]^{1-\alpha}} d\mu(y).$$

is bounded from $L^p(\mu)$ into $L^q(\mu)$.

Theorem 3.2. *Under the same assumption as that of Lemma 3.1, the conclusions of Theorem 1.10 and Theorems 1.11 hold true, if I_α therein is replaced by T_α .*

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